Conformal Transformations Combined with Numerical Techniques, with Applications to Coupled-Bar Problems

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Abstract—This paper describes a new approach to the solution of two-dimensional boundary value problems which eliminates the disadvantages and combines the advantages of both conformal transformations and numerical methods. The conformal transformations are used to remove potential gradient singularities, and numerical (e.g., finite difference) methods may then be applied to the resulting almost-regular field problems. Boundary value problems previously regarded as very difficult become tractable, and considerable savings in computer time and storage requirements are achieved. The method is applied to the calculation of the even and odd mode capacitances of cylindrical rods between plane parallel ground planes. Excellent agreement with results obtained previously is demonstrated.

I. Introduction

Two-dimensional boundary value problems occur frequently in many branches of microwave engineering, and similar problems with identical mathematical formulations are common in practically every branch of physics and engineering. This paper is restricted to solutions of the Laplace equation

$$\nabla^2\phi=0. \tag{1}$$

Numerical methods of solution have become popular because of the widespread availability of computers. Comprehensive reviews of finite difference, finite element, and other methods of discretization have been given in the various special issues of this Transactions on computer-oriented microwave practices [1], [2]. The review papers give an account of the advantages and disadvantages of the various approaches, although it is still difficult or even impossible to give estimates of accuracy, or to know *a priori* how fine a discretization to use to specify a given accuracy [3]. It is probably fair to state that it is difficult also to decide on the best choice of method for a given problem. One of the reasons for the difficulties encountered is that the finite difference formula is not applicable rigorously to re-entrant conductor corners, since the potential gradient is singular and Taylor's theorem is invalid. Similar problems occur with other numerical methods. Some techniques require large computer storage, and/or are very slow, usually because of convergence problems.

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An alternative approach which is more desirable is to use the conformal transformation technique. This is purely algebraic and does not involve any large scale numerical operations. Hence the method is very fast, in fact practically instantaneous using computers. Although more and more problems are proving amenable to the conformal transformation approach, e.g., [4], [5], the usual assumption is that it may be applied only to a very restrictive range of problems. These arise when the boundaries can be transformed into simple straight lines or circles, leading eventually to a regular field pattern having an exact known solution. It is the object of this paper to point out that conformal transformations have a far greater range of applicability than previously implied, and that conformal transformations may be combined with numerical methods to eliminate the difficulties of a purely numerical approach.

II. The Combined Conformal Transformation/Numerical Method

The basic approach is to eliminate all singularities in the potential by a conformal transformation in order to obtain a final set of boundaries which are as near to perfect regularity as possible, and then to apply numerical techniques to the final boundaries. Since the field is almost regular a rather coarse mesh may be used if a finite difference method is to be employed in the final step. It is necessary to keep careful track of all nonregular boundaries, and the method then remains exact, at least up to the step where a numerical method must be finally adopted. There is no need to search for a transformation giving exactly regular boundaries, so that most practical problems are amenable to treatment in this way.

One of the first published applications of the technique appears to be by Chang [6], but this short paper does not describe the wide generality of the method, and its full implications may not be clearly apparant. It does not point out that fairly precise numerical techniques may be combined with the conformal transformation to take advantage of the best features of each approach, which were formerly applied separately.

Another application of the technique is to the solution of open structures where the boundaries extend to infinity, as described by Decréton *et al.* [7], [8]. Here we may regard the point at infinity as a singularity, and resolve the difficulty by a conformal transformation which results in an equivalent closed-boundary problem.

III. Example: Coupled Bars of Circular Cross Section

The technique is best illustrated by means of an example, and the one chosen here is the coupled cylindrical bar configuration shown in Fig. 1(a). This simplifies to the solution of the even and odd mode boundary value problems shown in Fig. 1(b). A related (but not identical) coupled bar problem was solved by Cristal [9] using an integral representation of the field potential, followed by numerical discretization to solve the integrals. The even and odd mode configurations of [9] have electric or magnetic walls symmetrically located on either side of the bar, rather than the asymmetric configurations of Fig, 1(b). It will be shown here that solutions of Fig. 1 lead to solutions for the case of the multiple coupled-bar array also, and consequently are as useful as those given in [9].

No exact conformal transformation is known for the boundaries shown in Fig. 1(b), but the outer boundary walls $ABCDE$ may be transformed to a perfect circle and the inner circular boundary to a roughly circularly shaped conductor by means of the transformation:

$$w=\frac{\tanh(z-a)}{\tanh(z+a)}. \tag{2}$$

A similar type of transformation was used by Wheeler [10] to solve the simpler problem of a single bar between parallel ground planes, and (2) was derived logically from this. In order to indicate practical realistic results, the conformal transformations will be drawn to scale for the case $D/b=0.5$, $s/b=0.25$, and the w-plane is given for this case in Fig. 2. The equation of the outer circle corresponds to $z=x\pm j\pi/4$ for portions AB and ED of Fig. 1(b) and to $z=jy$ for BCD, and these regions all lie on $|w|=1$ in the w-plane. Points B and D are given by $z=\pm j\pi/4$ and are located as indicated in Fig. 2. The equation of the inner conducting boundary is given by substituting the polar equation for the circle in Fig. 1(b), i.e.,

$$z=a+r\cos\phi+jr\sin\phi \tag{3}$$

in (2), giving the following locus in polar coordinates:

$$|w_1|=\left\{\frac{\tanh^2u+\tan^2v}{1+\tanh^2u\tan^2v}\Big/\frac{\tanh^2u'+\tan^2v}{1+\tanh^2u'\tan^2v}\right\} \tag{4}$$

$$\arg w_1=\tan^{-1}\left\{\frac{\tan v}{1+\tan^2v}\cdot\frac{1-\tanh^2u}{\tanh u}\right\}-\tan^{-1}\left\{\frac{\tan v}{1+\tan^2v}\cdot\frac{1-\tanh^2u'}{\tanh u'}\right\} \tag{5}$$

where

$$u=r\cos\phi$$
$$v=r\sin\phi$$
$$u'=(2a+r)\cos\phi. \tag{6}$$

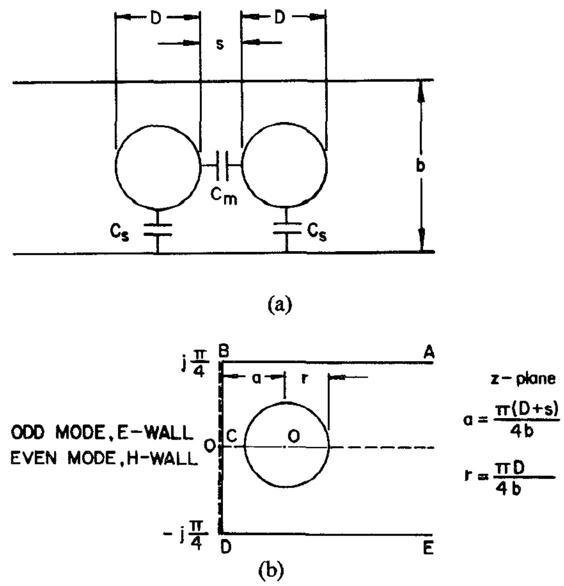

Fig. 1. (a) Cross section through coupled cylindrical bars. (b) Even- and odd-mode problems.

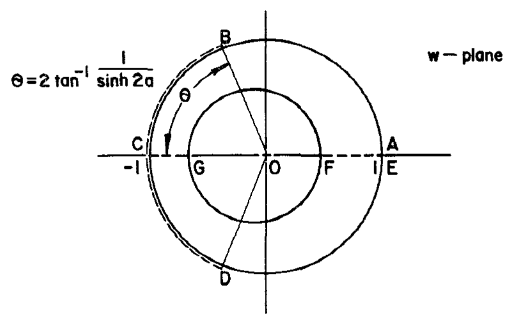

Fig. 2. First transformation, $D/b=0.5$, $s/b=0.25$.

A second conformal transformation

$$t=-\ln w \tag{7}$$

is now applied to the upper-half w-plane giving the almost regular boundaries (at least for the odd mode) of Fig. 3. For convenience the t-plane has been rotated through 90°. The equation of the smooth curve GF is

$$t_1=\arg w_1+j\ln|w_1| \tag{8}$$

with $|w_1|$ and arg w_1 being given by (4)–(6). Since (by symmetry) tangents to the curve GF at G and F form a right angle to GC and FA, respectively, and the GF boundary varies smoothly and gradually, the field is highly regular, and the odd-mode capacitance can be written down approximately but very accurately in terms of the line integral of the flux along the boundary AC. Let a potential of 1 V be applied to GF with the base conductor AC at zero potential. Hence at any point on AC corresponding to a vertical ordinate of GF equal to h, the gradient of the potential on AC directed normally to this boundary is simply

$$\nabla\phi=1/h. \tag{9}$$

Hence the capacitance between GF and AC, equal to half

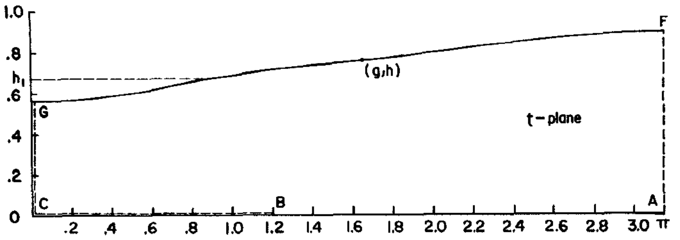

Fig. 3. Second transformation, $t=-\ln w$.

of the required odd-mode capacitance C_{oo}, is given by

$$\frac{C_{oo}}{\varepsilon}=2\int_0^{\pi}\nabla\phi\cdot dg=2\int_0^{\pi}\frac{dg}{h}$$

where dg is a line increment parallel to the boundary AC, ε is the free space permittivity, so that C_{oo}/ε is given in its conventional normalized form. In the case of $D/b=0.5$ and $s/b=0.25$ shown drawn to scale in the figures, equation (10) results in $C_{oo}/\varepsilon=8.77$. A precise finite difference calculation with the usual relaxation procedure using a stationary integral expression for the capacitance yields a value of $8.801\pm.002$, i.e., the error in using (10) is less than 0.4 percent.

At this stage several points may be noted. Firstly the precise value quoted above was obtained using a quite coarse grid with only about 100 meshes (the accuracy was confirmed both by varying the mesh size and by forming both upper and lower bounds). Secondly the simple expression (10) results always in a lower bound to the exact solution, since in practice the electric charge will tend to concentrate near points G, C in Fig. 3, giving a higher capacitance. Thirdly, the approximate value may be sufficiently exact in practice, a point to be discussed further below. Occasionally for some problems an approximate closed form solution may be obtained.

Of course not all problems will be amenable to final boundaries shaped as advantageously as shown in Fig. 3, but the general technique should enable the conformal transformation approach to be applied more freely than hitherto. Some extra mathematical effort is required both to find and then apply the appropriate transformations, this in comparison to plunging directly into a numerical solution.

Returning to the coupled-bar problem, the even mode capacitance, corresponding to the magnetic-wall BC of Fig. 3, has yet to be found. Rather than carrying out a finite difference calculation for the even mode, it is simpler to make yet another conformal transformation to the t' plane of Fig. 4, where

$$t'=\frac{h_1}{\pi}\cosh^{-1}\left(\frac{\cosh(\pi t/h_1)-\sinh^2(\pi\theta/2h_1)}{\cosh^2(\pi\theta/2h_1)}\right) \quad (11)$$

where h_1 is an arbitrary point on GF, and is chosen here as shown in Fig. 3. Equation (11) is derived by mapping the positive-half straight lines $h=0$ and $h=h_1$ together with the line connecting $(0,h_1)$ and $(0,0)$, onto the upper half plane using the appropriate Schwartz–Christoffel transformation, i.e., the right angles at $(0,0)$ and $(0,h_1)$ are

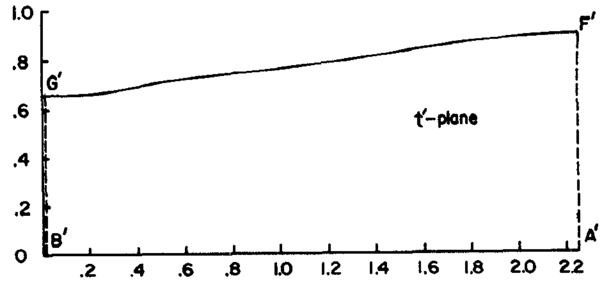

Fig. 4. t'-plane for calculation of even-mode capacitance.

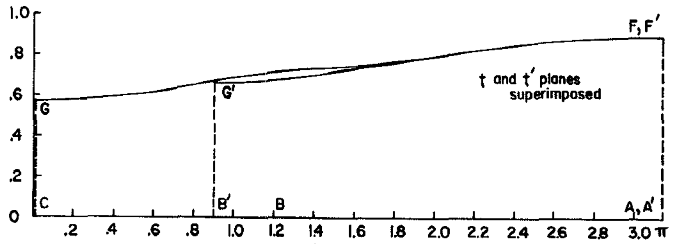

Fig. 5. Even- and odd-mode boundaries superimposed, to scale.

"unfolded". This figure is then refolded into the original shape by applying the reverse transformation, but with one of the right angles now corresponding to point B, becoming B' in Fig. 4. Now the t'-plane problem is identical to the t-plane odd-mode problem, and is solved in the same way as described above. The result for the even mode capacitance for our $D/b=0.5, s/b=0.25$ case is $C_{oe}/\varepsilon=5.79$ by the simple flux integral method of (10), or 5.812 ± 0.002 by the finite difference method. The mutual capacitance C_m of Fig. 1(a) is given by

$$\frac{C_m}{\varepsilon}=\frac{C_{oo}-C_{oe}}{2\varepsilon}=\begin{matrix}1.49 \text{ (approximate method)}\\ 1.4945\pm0.002 \text{ (precise method)}\end{matrix}. \quad (12)$$

It is instructive to superimpose the odd and even mode boundaries of Figs. 3 and 4 on the same diagram with the FA, $F'A'$ magnetic walls coinciding as shown in Fig. 5. This illustrates the obvious fact that the even and odd mode fields tend to coincide in regions far removed from the coupling region. The difference capacitance (mutual coupling) is clearly defined, and the problem retains its physical significance throughout.

Although (by symmetry) $A'F'$ is normal to both $F'G'$ and $A'B'$, F' is not *exactly* vertically above A', i.e., the magnetic boundary $A'F'$ possesses slight curvature. In general the distance $A'B'$ (which maybe defined as the even mode base) is less than the horizontal F' coordinate. For the example illustrated the difference is 2×10^{-5}, which is quite negligible. The effect becomes more pronounced as the bar diameter is reduced, as expected from physical considerations, but need not be taken into account until $D/b<0.25$. For example if $D/b=0.2$, $s/b=0.1$, the even mode base is 1.3398, but the F' coordinate is 1.3549, a 1-percent difference. The effect is less pronounced for looser coupling, i.e., if $D/b=0.2$, $s/b=1.0$, the even mode base is 2.9881 and the F' coordinate is 2.9909, a 0.1-percent difference.

The greatest error in using the approximate integral (10) rather than a precise numerical procedure occurs for very tightly coupled bars, e.g., when $s/b \cong 0.1$ corresponding to mutual capacitance values in the 3–4 range, but the errors are usually less than 2 percent. Accuracy of this order is sufficiently good for filter designs (where the majority of applications occur), and avoiding more precise methods results in smaller and faster computer programs.

The application of these results to the solution of the problem of multiple-coupled bars is given in Appendix II.

IV. Further Discussion of Accuracy

The usual way of testing the accuracy of a new boundary-value technique is to apply it to problems having exact solutions obtained by other methods. This is not possible here because all such problems automatically give a precisely correct result, since an exact conformal transformation will exist. Hence it is necessary to test it against previously published results for "real" problems. The coupled-rod problem may be checked roughly against Cristal's results [9], (see Appendix II), but more precisely against those of Chisholm [11] for the odd-mode case, and Wheeler [10] for the case of a single uncoupled bar.

The latter problem may be derived from Fig. 1 by allowing $a \to \infty$, and this limiting case is easily incorporated into the general coupled-bar program. Changing the origin of Fig. 1(b) to the bar center by replacing z by $z+a$ and allowing $a \to \infty$, (2) reduces to

$$w = \tanh z. \tag{13}$$

However, as stated above, it is unnecessary to program this limiting case separately. Wheeler [10] treats two cases very precisely, namely those for impedances of 50 Ω and 19.318 Ω, corresponding to $D/b = 0.548959$ and $D/b = 0.866025$. Wheeler takes the value of free space impedance as 120π, and later corrects for the actual velocity of light and dielectric constant of air, but these considerations are immaterial here. The normalized capacitances for the above cases become 7.53982 ± 0.00045 and 19.5150 ± 0.04, with the tolerances being those quoted by Wheeler.[1] The low impedance case is a very severe test of any theory, and the estimated accuracy is therefore worse, but still within 0.2 percent. The very good quoted accuracies are due to the use of a more sophisticated conformal transformation than that given by (13), one which gives an even more regular field, from which a power series expansion for the characteristic impedance was determined [10]. The simpler method used here would be expected to give slightly less accurate results.

The conformal transformation (13) gives an almost-regular problem similar to that of Fig. 3. The 50-Ω problem gives minimum and maximum ordinates of 0.776495 and 0.900705. Actually only one-quarter of the capacitance need be calculated in the isolated bar case, i.e., the base is 0.5π rather than π. The mesh pattern is defined as

$$(\text{no. of base grids}) \times (\text{no. of grids in minimum ordinate}).$$

[1]The accuracies in this early paper [10] may be overstated.

With a mesh of 10×5, $C/\varepsilon = 7.5381$, and with one of 20×10, $C/\varepsilon = 7.5367$. These values are within 0.04 percent of Wheeler's figure of 7.5398. The simple line integral of (10) gives $C/\varepsilon = 7.49$, low by 0.66 percent.

In the finite difference method the capacitance is calculated using the stationary integral technique described in many previous papers [1], i.e.,

$$\frac{C}{\varepsilon} = \int\int (\nabla\phi)^2 \cdot dA \tag{14}$$

where the double integral is taken over the entire area A. This is not only more accurate than the line integral (10) but also converges must faster in the relaxation process. Thus the 10×5 mesh above converged to 5 decimal place accuracy in 16 iterations. The line integral converges to almost the same result in about 60 iterations (without using an acceleration factor).

In the low impedance case above with $D/b = 0.86605$, the minimum ordinate is 0.212018 and the maximum is 0.524869. The simple line integral (10) gives $C/\varepsilon = 19.24$, the finite difference method with a mesh of 16×2 gives 19.444, 20×3 gives 19.471, 25×8 gives 19.487, and 30×10 gives 19.491. The finite difference value converges to approximately 19.493, which is 0.1 percent lower than Wheeler's figure of $19.515 \pm .04$, but within his quoted margin of error.

The mesh sizes used to obtain these results are very coarse by previous standards, where meshes of the order 100×100 have been found necessary to give reasonable accuracy. The number of nodes is reduced by two orders of magnitude in the above examples.

Turning now to the coupled bar case of Fig. 1, the odd mode capacitance problem was solved by Chisholm using a variational technique [11]. The method was extended to the even mode case by McDermott [12], and the results are given in Appendix I. A comparison between a number of results obtained from these formulas and the method described in this paper is given in Table I. The first ten results are for cases where odd mode impedance is 50 Ω, and the even mode impedance varies. The next eight are for $D/b = 0.4$ with varying s/b. The maximum difference between the results occurs for $D/b = 0.4$, $s/b = 0.08$, i.e., a case where the coupling is extremely tight. For most cases the agreement is within 0.2 percent.

The results in the right hand columns were obtained using square meshes with dimensions of the order of 0.2×0.2. Rather than using the finite difference method with relaxation, a resistive analogue method with exact matrix inversion (i.e., no relaxation) was used. This is slightly less accurate than the normal finite difference method, but the discrepancy is usually in the third decimal place. This technique is beyond the scope of this paper, and will be described in a future publication [16].

V. Application to Other Problems

Over the course of several years the method has been applied to other problems involving coupled bars, including for example coupling via thick symmetric or asymmetric irises. Three of these problems are illustrated in Fig. 6.

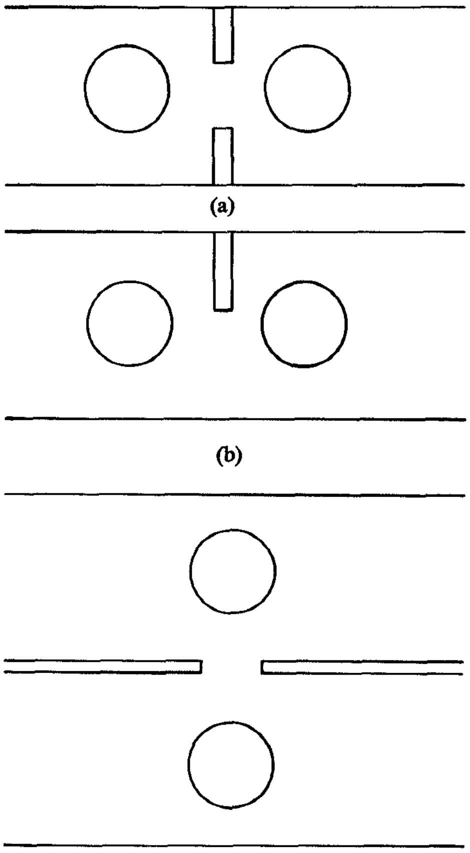

Fig. 6. Illustrating other problems solvable by the combined conformal transformation/numerical method.

TABLE I
COMPARISON BETWEEN NORMALIZED CAPACITANCES C/ε OBTAINED FROM THE CHISHOLM/McDERMOTT METHOD AND THIS PAPER

D/b	s/b	CHISHOLM / Mc DERMOTT		THIS WORK	
		Even	Odd	Even	Odd
.354	.176	3.9142	7.5347	3.9153	7.5528
.400	.226	4.5093	7.5347	4.5080	7.5404
.436	.280	5.0281	7.5347	5.032	7.5501
.462	.338	5.4731	7.5347	5.4718	7.5371
.482	.398	5.8497	7.5347	5.8446	7.5339
.498	.462	6.1648	7.5347	6.1721	7.5436
.510	.528	6.4257	7.5347	6.4380	7.5514
.518	.596	6.6404	7.5347	6.6424	7.5371
.534	.806	7.0727	7.5347	7.0710	7.5359
.544	1.168	7.3855	7.5347	7.3806	7.5285
.400	.080	4.1646	11.0783	4.1553	11.2882
.400	.120	4.2631	9.4595	4.2626	9.5317
.400	.160	4.3578	8.4935	4.3516	8.5419
.400	.200	4.4483	7.8478	4.4444	7.8652
.400	.240	4.5340	7.3863	4.5358	7.3982
.400	.400	4.8273	6.3903	4.8263	6.3914
.400	.600	5.0826	5.8862	5.0819	5.8848
.400	.760	5.2134	5.6949	5.2130	5.6946

In each case the thick iris retains most of its main geometric features through the conformal transformations, and finally one of the thick-iris formulas to be found in the literature may be applied, or further conformal transformations used to eliminate the right angles of the iris. The configuration shown in Fig. 6(c) is frequently used to realize cross-coupling in linear phase filters.

It will be noted that treatment of these problems involving thick irises would be very difficult using a purely numerical method, since a very fine mesh would be required in the region of the irises. The *thin* symmetrical iris case has been solved by Cristal [13] using the finite difference method. If exact solutions to such problems exist, they require at least third ordered elliptic functions, since there are usually several right-angle discontinuities. In the method of this paper the problem is taken in stages, with some right angles unfolded in an initial transformation and the remainder in a final transformation. The final boundaries are irregular but smooth, and the functions are elementary.

Configurations with mixed dielectrics are easily handled by keeping track of the locus of the dielectric boundary (e.g., [7], [8]).

VI. Conclusions

Boundary value problems may be simplified and made well-conditioned using conformal transformations to give almost regular field patterns. The numerical work is simplified, i.e., the number of nodes may be 100 times less than that required for a conventional numerical solution, while the accuracy is extremely good. Some problems which are beyond the limits of either purely numerical techniques or pure conformal transformations become tractable using the combined technique. The method is much faster than the numerical technique alone, and may be used in an iteration procedure which determines dimensions from given capacitances, i.e., it is useful in synthesis as well as analysis.

The method may be used to treat complicated boundaries having several right angles without recourse to higher order functions, and to mixed dielectric problems.

Appendix I
Coupled-Bar Equations of Chisholm [11] and McDermott [12]

The results given here are expressed in terms of Chisholm's nomenclature, which is shown in Fig. 7. The results are given in a power series in b/a as far as $(b/a)^4$. The even or odd mode normalized capacitance C/ε is given by

$$\frac{\varepsilon}{C}=\left[X^{0,0}+2\alpha_1X^{0,1}+2\alpha_2X^{0,2}+2\alpha_1\alpha_2X^{1,2}+\alpha_1^2X^{1,1}+\alpha_2^2X^{2,2}\right] \quad (15)$$

where

$$\alpha_1=-\begin{vmatrix}X^{1,0} & X^{1,2}\\ X^{2,0} & X^{2,2}\end{vmatrix} \Big/ \begin{vmatrix}X^{1,1} & X^{1,2}\\ X^{2,1} & X^{2,2}\end{vmatrix} \quad (16)$$

$$\alpha_2=-\begin{vmatrix}X^{1,1} & X^{1,0}\\ X^{2,1} & X^{2,0}\end{vmatrix} \Big/ \begin{vmatrix}X^{1,1} & X^{1,2}\\ X^{2,1} & X^{2,2}\end{vmatrix}. \quad (17)$$

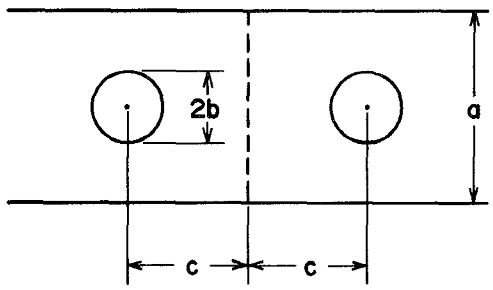

Fig. 7. Chisholm's nomenclature for coupled bars.

The $X^{j,k}$ differ for the two modes, are given as follows, with the upper sign giving the odd-mode capacitance and the lower sign the even-mode capacitance:

$$X^{0,0}=\frac{1}{2\pi}\left[\ln\frac{2a}{\pi b}\pm\ln\tanh\frac{\pi c}{a}\right] \quad (18)$$

$$X^{1,0}=X^{0,1}=\mp\frac{1}{4}\left[\operatorname{cosech}\frac{2\pi c}{a}\right]\cdot\left(\frac{b}{a}\right) \quad (19)$$

$$X^{1,1}=\frac{1}{8\pi}-\frac{\pi}{8}\left[\frac{1}{6}\pm\coth\frac{2\pi c}{a}\operatorname{cosech}\frac{2\pi c}{a}\right]\cdot\left(\frac{b}{a}\right)^2 \quad (20)$$

$$X^{0,2}=X^{2,0}=\frac{\pi}{8}\left[\frac{1}{6}\mp\coth\frac{2\pi c}{a}\operatorname{cosech}\frac{2\pi c}{a}\right]\cdot\left(\frac{b}{a}\right)^2 \quad (21)$$

$$X^{1,2}=X^{2,1}=\mp\frac{\pi^2}{64}\left[\tanh\frac{\pi c}{a}\operatorname{sech}^2\frac{\pi c}{a}+\coth\frac{\pi c}{a}\operatorname{cosech}^2\frac{\pi c}{a}\right]\cdot\left(\frac{b}{a}\right)^3 \quad (22)$$

$$X^{2,2}=\frac{1}{16\pi}-\frac{\pi^3}{256}\left[\frac{14}{15}\pm\left(\operatorname{cosech}^2\frac{\pi c}{a}\right)\cdot\left(2\coth^2\frac{\pi c}{a}+\operatorname{cosech}^2\frac{\pi c}{a}\right)\pm\operatorname{sech}^2\frac{\pi c}{a}\left(2\tanh^2\frac{\pi c}{a}-\operatorname{sech}^2\frac{\pi c}{a}\right)\right]\cdot\left(\frac{b}{a}\right)^4. \quad (23)$$

Upper sign for odd mode, lower sign for even mode.

Appendix II
Multiple-Coupled Bar Arrays

The general case of multiple-coupled bars may be treated by a similar put possibly rather more convenient method than that of Cristal [9], in the sense that graphical interpolation will be avoided. The method to be described here parallels that for rectangular coupled bars [4], to the extent that the equivalents of parallel-plate and fringing capacitances will be defined for circular bars. Fig. 8(a) shows an isolated uncoupled bar having equivalent parallel plate capacitance C_g and fringing capacitances C_f, so that the total capacitance in this case is

$$C_g+C_f=C_I \quad (24)$$

when two such bars are coupled the odd- and even-mode circuits of Fig. 8(b) and (c), respectively, result, where

$$C_g+C_f+2C_m=C_{oo} \quad (25)$$

$$C_g+C_f=C_{oe}. \quad (26)$$

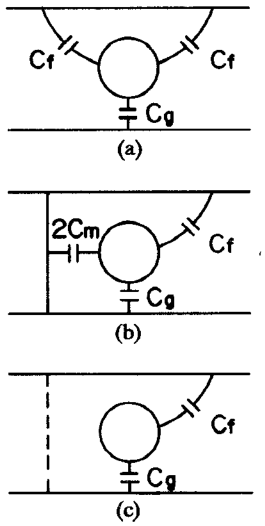

Fig. 8. Definition of "equivalent-parallel plate" and "fringing" capacitances for cylindrical bars.

It is not necessary to define distinct even and odd mode fringing capacitances in the circular bar case, these being incorporated into the single mutual capacitance C_m. The three equations (24)–(26) may be solved to give

$$C_m = \tfrac{1}{2}(C_{oo} - C_{oe}) \tag{27}$$

$$C_f = C_I - C_{oe} \tag{28}$$

$$C_g = 2C_{oe} - C_I. \tag{29}$$

Hence the definition of the equivalent parallel plate and fringing capacitances requires a solution for the isolated bar capacitance C_I, as described in the main text.

Comparison with Cristal's results [9] may be obtained by forming the symmetrical odd and even mode circuits of Fig. 9, which indicates that C_g and C_m are identical to the quantities defined and plotted in [9]. In our $D/b = 0.5$, $s/b = 0.25$ example, $C_I = 6.75$, $C_{oo} = 8.77$, $C_{oe} = 5.79$, giving $C_g = 4.83$, $C_m = 1.49$, $C_f = 0.96$.

In the case of an array of unequal bars where the self and mutual capacitances are given, the dimensions are found as follows. Assuming that the end bar is decoupled from an end wall, its self capacitance is $C_g + C_f = C_{oe}$. Choosing any value of D/b, there is one and only one value of s/b giving the required mutual capacitance, and this is found by iteration. The chosen D/b is correct only if it gives the correct self-capacitance, and an overall iteration must be carried out to find this correct value of D/b. Hence there is one iteration loop nested inside another, but forming appropriate slopes the Newton–Raphson technique will converge very rapidly.

In the case of an inner bar, the left and right hand values are dealt with separately as described in [9]. The known self capacitance here is simply C_g, and again the unique D/b and s/b values are found. Then the mean of the left and right D/b values is chosen as the actual D/b for that bar. The s/b values are formed similarly as arithmetic means.

The iteration process is sufficiently fast for the method to be used in a routine program for calculation of the dimensions of a large array, i.e., no prestored results are used. By carrying out simple capacitance matrix transformations [15] equivalent arrays with equal bar diameters are derived using no more than three overall iterations.

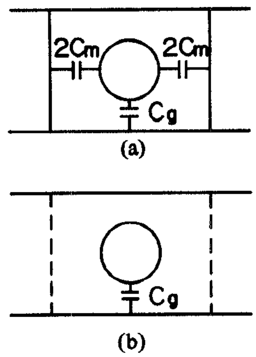

Fig. 9. (a) Symmetrical odd-mode configuration. (b) Symmetrical even-mode configuration.

The results obtained from this procedure are almost identical to those given in [9] for values of $D/b > 0.2$, a condition which should hold in almost all practical cases. Smaller values of D/b imply the possibility of coupling between nonadjacent bars, and structures of such high impedance would normally be avoided.

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